F. anal. 10. 3. 2025

- slides, info (ref grp)

A. Intermezzo: Interchanging limits and int's in $\mathbb{R}$

- Similar to results for $[a,b]$
- Recall: $C_m(\mathbb{R}) = \{ f \in C(\mathbb{R}) : |f(x)| \leq (1+|x|^2)^{-\frac{1+\varepsilon}{2}} \}$

  $f \in C_m \Rightarrow f$ unif. cont. ; $\int_{\mathbb{R}^d} f$, $\int_{\mathbb{R}^d} |f|$ exist

Def. I: a) $f_n \to f$ <u>locally uniformly</u> if

$$\| (f_n - f)(x) \chi_{|x|<N} \|_\infty \xrightarrow[n\to\infty]{} 0 \quad \forall N > 0.$$

Ex. I: $\min(x^2, n) \xrightarrow[n\to\infty]{} x^2$ loc. unif. in $\mathbb{R}$ (but not unif.)

Lem. I: ("Dominated conv." in $C_m(\mathbb{R})$)

If $f_n, f \in C_m(\mathbb{R})$,

(i) $f_n \to f$ loc. unif.

(ii) $\exists\, \tilde{g} \in C_m(\mathbb{R})$ s.t. $|f_n(x)| \leq \tilde{g}(x)$

(uniform tail control)

then $\lim\limits_{n\to\infty} \int_{\mathbb{R}} f_n(x)\,dx = \int_{\mathbb{R}} f(x)\,dx$

Pf.:

$$\left| \int f_n - \int f \right| \leq \underbrace{\int_{|x|>N} (|f_n| + |f|)}_{\overset{(ii)}{\leq} \int_{|x|>N} (|\tilde{g}| + |f|) = O(N^{-\varepsilon}) \longrightarrow 0,\ N\to\infty\ (f, g \in C_m)} + \int_{|x|\leq N} \underbrace{\| (f_n - f) \chi_{[-N,N]} \|_\infty}_{\xrightarrow[(i)]{} 0,\ n\to\infty\ (N \text{ fixed})} dx \xrightarrow[n\to\infty]{} 0$$

$\square$

Cor. 1: Assume

(A1) $f, \frac{\partial f}{\partial t} \in C([a,b] \times \mathbb{R})$

(A2) $\exists\, \tilde{g} \in C_m(\mathbb{R})$ s.t. $|f(t,x)| + |\frac{\partial f}{\partial t}(t,x)| \leq \tilde{g}(x)$

Then
$$\frac{d}{dt} \int_{\mathbb{R}} f(t,x)\, dx = \int_{\mathbb{R}} \frac{\partial f}{\partial t}(t,x)\, dx.$$

"Pf.:" $D_h f \to \frac{\partial f}{\partial t}$ loc. unif., $|D_h f| \leq \tilde{g}$, use Lem 1. □

Cor. 2: If $f \in \underset{\text{bnd.}}{C_m}(\mathbb{R})$; $g, \partial_x g \in C_m(\mathbb{R})$

$$\partial_x (f * g)_x \in C_m(\mathbb{R}) \text{ and } \partial_x (f * g) = f * (\partial_x g)$$

"Pf.": $D_h(f*g) = f * D_h g$, use Cor 1. [Exercise 8] □

and note $\| f * \partial_x g \|_\infty \leq \| f \|_\infty \cdot \| \partial_x g \|_1$ □

$C_m * C_m \overset{\text{prev.}}{\subset} C_m$

Applications to PDEs

B. Heat eq'n on $\mathbb{R}$

$$\begin{cases} (1) & u_t = u_{xx} \text{ in } \mathbb{R} \times (0,\infty), \\ (2) & u(x,0) = f(x), \text{ in } \mathbb{R} \end{cases}$$

Prev. courses:

$$(1),(2) \xrightarrow{\mathcal{F} \text{ in } x} \hat{u}_t = (2\pi i \xi)^2 \hat{u} = -4\pi^2 \xi^2 \hat{u}, \quad \hat{u}(\xi,0) = \hat{f}$$

$$\underset{\text{solve ODE}}{\Longleftrightarrow} \hat{u}(\xi,t) = \hat{f}(\xi)\, e^{-4\pi^2 \xi^2 t} \quad \text{(chk!)}$$

$$\xrightarrow[\mathcal{F}^{-1}]{} u(x,t) = (f * \mathcal{H}_t)(x), \text{ where}$$

(3) $$\mathcal{H}_t(x) = \mathcal{F}^{-1}[\underbrace{e^{-4\pi^2\xi^2 t}}_{=\hat{K}_{4\pi t}(\xi)}] = K_{4\pi t}(x)$$

$$\overset{\text{def } K_\delta}{=} \frac{1}{(4\pi t)^{\frac{1}{2}}} e^{-\frac{x^2}{4t}}$$

Not rigorous! We now chk. that u solves (1)-(2):

Thm. 1: $f \in C_m(\mathbb{R})$; $\mathcal{H}_t$ def. by (3). Then

(a) $u(x,t) = f * \mathcal{H}_t(x) \in C_m^\infty(\mathbb{R}\times(0,T\infty))$

(b) u solves (1)

(c) $u(x,t) \xrightarrow[t\to 0]{} f(x)$ unif. and $u \in C_m(\mathbb{R}\times[0,T))$

Pf.:

~~(a) Chk: $\partial_x^k \partial_t^m \mathcal{H}_t(x) = R_{k,m}(x,t)\,\mathcal{H}_t(x)$, for $R_{k,m} \in C^\infty$ $(t>0)$~~

~~s.t. $|R_{k,m}| \le C(1+t^{-k-2m})(1+x^{k+2m})$.~~

~~Hence $\partial_x^k \partial_t^m \mathcal{H}_t \in S(\mathbb{R}) \subset C_m(\mathbb{R})$, and by Cor. 2,~~

~~(b) Since $S(\mathbb{R}) \subset C_m(\mathbb{R})$, by Cor. 2,~~

~~$\partial_x^k \partial_t^m u = f * (\partial_x^k \partial_t^m \mathcal{H}_t) \in C_m(\mathbb{R})$ for $t>0$.~~

(a), (b) $f, \mathcal{H}_t \in C_m \overset{\text{prev. res.}}{\Longrightarrow} u = f * \mathcal{H}_t \in C_m(\mathbb{R})$, $t>0$

Take F-transf. in x:

$$\hat{u} = \hat{f}\cdot\hat{\mathcal{H}}_t \;;\; \hat{\mathcal{H}}_t = e^{-4\pi^2\xi^2 t},\quad \hat{f} \in C_0(\mathbb{R})$$

Since $\hat{u}(t), \hat{u}(t) \in C_m(\mathbb{R})$ for $t > \overset{(\text{chk})}{0}$, by F-inv.:

$$(4) \quad \underbrace{u(x,t)}_{= \mathcal{F}^{-1}[\hat{u}(\cdot,t)](\xi)} = \int_{\mathbb{R}} \underbrace{\hat{f}(\xi) \cdot \widehat{\mathcal{H}}_t(\xi)\, e^{2\pi i x \xi}}_{F(\xi, x, t) \text{ cont.}} d\xi$$

9:00 Obs: $\underbrace{|\partial_x^k \partial_t^m F(\xi, x, t)|}_{} \leq C_{k,m}^{(t)} e^{-\xi^2 t}, \in C_m(\mathbb{R})$ for $t > 0$ fixed,

so Cor. 1 gives for $t > 0$,

$$\partial_x^k \partial_t^m u = \int_{\mathbb{R}} \hat{f}(\xi)(\partial_t^m \widehat{\mathcal{H}}_t(\xi))(\partial_x^k e^{2\pi i x \xi})\, d\xi,$$

Since k, m arbitrary, $u \in C^\infty(\mathbb{R} \times (0,\infty))$,

and

$$(\partial_t - \partial_x^2)\, u = \int \hat{f}(\xi)(\partial_t - \partial_x^2)\, \widehat{\mathcal{H}}_t e^{2\pi i x \xi} d\xi$$

$$= \int \hat{f}(\xi) \underbrace{(-4\pi^2 \xi^2 - (2\pi i \xi)^2)}_{=0} \widehat{\mathcal{H}}_t e^{2\pi i x \xi} d\xi$$

$$= 0$$

(c) $f * \mathcal{H}_t \underset{(3)}{\overset{\delta = 4\pi t}{=}} f * K_\delta \xrightarrow[t \to 0]{\text{prev. res.}} f$ unif. $\square$

$(\Rightarrow \delta \to 0)$

<u>Rem. 1</u>: (i) (a) + (c) $\Rightarrow$ $u \in C(\mathbb{R} \times [0,\infty))$. $(\cap\, C^\infty(\mathbb{R} \times (0,\infty)))$

(ii) (4) Fourier-integral representation of u (usefull!)

C. Poisson summation formula

Periodizations of $f \in S(\mathbb{R})$:

1) $F_1(x) = \sum_{n=-\infty}^{\infty} f(x+n)$

2) $F_2(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{2\pi i n x}$ $\hat{f}(n) = \mathcal{F}[f](n)$

Obs 1:

a) 1-per.: $F_1(x+1) = \sum_{n\in\mathbb{Z}} f(x+1+n) \overset{m=n+1}{=} \sum_{m\in\mathbb{Z}} f(x+m) = F_1(x)$

b) $f \in S(\mathbb{R}) \Rightarrow \max_{|x|\le R} |f(x+n)| \le \underbrace{\sup_{x\in\mathbb{R}} \left(x^2 |f(x)|\right)}_{=:K<\infty} \frac{1}{(R+n)^2} \le \frac{K}{n^2}$

Weierstrass M-test $\Rightarrow$ $F_1(x)$ conv. abs. and unif. on $|x| \le R$, $\forall R > 0$

$\Rightarrow F_1$ cont. (and bnd.) [unif. lim. of cont. f'ns]

$F_2(x+1) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{2\pi i n(x+1)} \overset{}{\underset{e^{2\pi i n}=1}{=}} F_2(x)$

c) $\hat{f} \in S(\mathbb{R}) \Rightarrow F_2$ abs., loc. unif. conv. ; cont.

Poisson: $F_1 = F_2$

Thm. 1: Poisson summation formula

$$f \in S(\mathbb{R}) \Rightarrow \sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{2\pi i n x}$$

$$\overset{x=0}{\Longrightarrow} \sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$$

Pf.:

By def. of F-series, the F-coeff.

$$\widehat{F_2}(m) = \hat{f}(m), \quad m \in \mathbb{Z}.$$

While

$$\hat{F}_1(m) = \int_0^1 \underbrace{\Big(\sum_{n\in\mathbb{Z}} f(x+n)\Big)}_{F_1} e^{-2\pi i m x}\,dx$$

$$\overset{\text{termw. int.}}{\underset{\substack{(\Sigma \text{ conv.} \\ \text{unif.}}}{=}} \sum_{n\in\mathbb{Z}} \int_0^1 f(x+n)\, e^{-2\pi i m x}\,dx$$

$$\overset{y=x+n}{=} \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(y)\, \underbrace{e^{-2\pi i m (y-n)}}_{=e^{-2\pi i m y}}\,dy$$

$$= \int_{\mathbb{R}} f(y)\, e^{-2\pi i m y}\,dy = \hat{f}(m)$$

Hence $\hat{F}_1(m) = \hat{f}(m) = \hat{F}_2(m)$ $\forall m \in \mathbb{Z}$

$$\underset{\substack{\text{F-series} \\ \text{uniq. (prev.)}}}{\overset{F_1, F_2 \text{ cont.}}{\Longrightarrow}} \quad F_1 = F_2, \quad x \in \mathbb{R}$$

$\square$

Rem. 1:

a) F-coeff. of F_1: $\hat{F}_1(n) = \mathcal{F}[f](n)$

b) Result and pf. still holds if $f, \hat{f} \in C_m(\mathbb{R})$.

Plan:
special f'ns
heat kernel
Heissenberg

F. anal. 13.3.2025

- Slides, info (ref g.p.)

The Poisson summation formula

A. θ, ζ, Γ func'ns

Theta f'n: $\theta(s) := \sum_{n\in\mathbb{Z}} \underbrace{e^{-\pi n^2 s}}_{= \hat{K}_{n^2}(s)}$ for $s > 0$

Obs: $f(x) = e^{-\pi s x^2} (= \hat{K}_s(x)) \Rightarrow \hat{f}(\xi) = s^{-\frac{1}{2}} e^{-\frac{\pi \xi^2}{s}} (= K_s(-\xi))$

Poisson's sum. formula:

$$(1) \quad \boxed{\theta(s) \overset{\text{def } \theta}{=} \sum_{n\in\mathbb{Z}} f(n) \underset{f\in S(\mathbb{R})}{\overset{\text{Poisson}}{=}} \sum_{n\in\mathbb{Z}} \hat{f}(n) \overset{\hat{f}=\ldots}{=} s^{-\frac{1}{2}} \theta(s^{-1})}$$

Zeta f'n: $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$ for $s > 1$

Important in number theory.

Gamma f'n: $\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\,dx$, $s > 0$

Obs: $\Gamma(s+1) = s\Gamma(s)$, $s > 0$; $\Gamma(n+1) = n!$ [Exercise 10]

Rel'n between θ, ζ, Γ:

$$(2) \quad \boxed{\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \frac{1}{2} \int_0^\infty t^{\frac{s}{2}-1} (\theta(t) - 1)\,dt \quad \text{for } s > 1}$$

Pf.: Exercise 10.

B. Heat kernels

(3) $\underset{\text{PDE}}{u_t = u_{xx}}$ in $(-\frac{1}{2},\frac{1}{2}) \times (0,\infty)$, $\underset{\text{I.C.}}{u(t=0) = f}$, $\underset{\text{B.C.}}{(u \text{ 1-per.})}$

(4) $v_t = v_{xx}$ in $\mathbb{R} \times (0,\infty)$, $u(t=0) = f$, $\underset{\text{"B.C."}}{(u \in C_m)}$

Previously:

$$u(x,t) = (H_t \underset{(0,1)}{*} f)(x), \qquad v(x,t) = (\mathcal{H}_t \underset{\mathbb{R}}{*} f)(x)$$

where

$$H_t(x) = \sum_{n\in\mathbb{Z}} e^{-4\pi^2 n^2 t} e^{2\pi i n x} \qquad \text{(Heat kernel on } \mathbb{S}\text{)}$$

$$\mathcal{H}_t(x) = \frac{1}{(4\pi t)^{\frac{1}{2}}} e^{-\frac{x^2}{4t}} \qquad \text{(Heat kernel on } \mathbb{R}\text{)}$$

Since $\hat{\mathcal{H}}_t(\xi) = e^{-4\pi^2 \xi^2 t}$, Poisson sum. formula gives:

Thm. I: $$H_t(x) \overset{\text{def } H, \hat{\mathcal{H}}}{=} \sum_{n\in\mathbb{Z}} \hat{\mathcal{H}}_t(n) e^{2\pi i n x} \underset{\underset{\mathcal{H}\in S(\mathbb{R})}{\uparrow}}{\overset{\text{Poisson}}{=}} \sum_{n\in\mathbb{Z}} \mathcal{H}_t(x+n)$$

H_t = periodization of $\mathcal{H}_t$!

Cor. I: H_t is a good kernel on $\underset{\substack{\text{is}\\ [-\frac{1}{2},\frac{1}{2}]}}{\mathbb{S}}$ as $t \to 0$.

Pf.:

(i) $$\int_{-\frac{1}{2}}^{\frac{1}{2}} H_t(x)\,dz \underset{\text{abs+unif.}}{\overset{\Sigma\text{ conv.}}{=}} \sum_{n\in\mathbb{Z}} e^{-4\pi^2 n^2 t} \underbrace{\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i n x}\,dx}_{\delta_{n0}} = e^0 \cdot 1 = \underline{\underline{1}}$$

(ii) $$H_t(x) = \sum_{n\in\mathbb{Z}} \underbrace{\mathcal{H}_t(x+n)}_{\geq 0} \geq 0$$

(iii) $$x \in [-\tfrac{1}{2}, \tfrac{1}{2}] : \quad H_t(x) = \mathcal{H}_t(x) + \underbrace{\sum_{|n|\geq 1} \mathcal{H}_t(x+n)}_{\mathcal{E}_t(x)}$$

$$|\mathcal{E}_t(x)| = \frac{1}{\sqrt{4\pi t}} \sum_{|n|\geq 1} \underbrace{e^{-(x+n)^2/4t}}_{\leq \leq \cdot e^{-n^2/16t}}$$

$$\left[-(x+n)^2 \overset{|x|\leq\frac{1}{2}}{\leq} -(|n|-\tfrac{1}{2})^2 \overset{|n|\geq 1}{\leq} -(|n|-\tfrac{1}{2}|n|)^2 = -\frac{n^2}{4}\right]$$

$$t\in(0,1): \quad \frac{n^2}{t} \geq n^2 \text{ and } \frac{n^2}{t} \geq \frac{1}{t}$$

$$\Rightarrow e^{-\frac{1}{16}\frac{n^2}{t}} \leq e^{-\frac{1}{16}(\frac{1}{2}n^2 + \frac{1}{2}\frac{1}{t})} \leq e^{-\frac{n^2}{32}} \cdot e^{-\frac{1}{32t}}$$

$$\Rightarrow |\xi_t(x)| \leq \frac{1}{\sqrt{4\pi}} \underbrace{t^{-\frac{1}{2}} \overbrace{e^{-\frac{1}{32t}}}^{e^{-\frac{1}{64t}} \cdot e^{-\frac{1}{64t}}}}_{\substack{s = t^{-\frac{1}{2}} \\ \leq \underbrace{\sup_{s>0}(s e^{-\frac{1}{64}s^2})}_{<\infty} e^{-\frac{1}{64t}}}} \underbrace{\sum_{|n|\geq 1} e^{-\frac{n^2}{32}}}_{<\infty} \leq \tilde{C} e^{-\frac{1}{64t}}$$

Hence

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |\xi_t(x)|\,dx \leq 1 \cdot \tilde{C} e^{-\frac{1}{64t}} \xrightarrow[t\to 0]{} 0$$

Claim: $$\int_{-\frac{1}{2}}^{\frac{1}{2}} |\xi_t(x)|\,dx \xrightarrow[t\to 0]{} 0$$

$$\int_{\eta<|x|<\frac{1}{2}} |H_t(x)| \leq \underbrace{\int_{|x|>\eta} |\mathcal{H}_t(x)|}_{\substack{\downarrow\ \mathcal{H}_t \text{ good} \\ \text{kernel in } \mathbb{R} \\ 0}} + \underbrace{\int_{-\frac{1}{2}}^{\frac{1}{2}} |\xi_t(x)|}_{\substack{\downarrow \text{Claim} \\ 0}} \xrightarrow[t\to 0]{} 0 \quad (\forall \eta \in (0, \tfrac{1}{2}))$$

<u>Pf. Claim</u>:

$\square$

11:04

By <u>results</u> for good kernels:

<u>Cor. 2:</u> $f \in C(\mathbb{S}) \Rightarrow u(x,t) = (H_t * f)(x) \xrightarrow[t\to 0]{} f(x)$ unif. in $\mathbb{S}$

C. Heisenberg's uncertainty principle

Thm. 2: If $\psi \in S(\mathbb{R})$, $\int_{\mathbb{R}} |\psi|^2 = 1$, then:

(a) $\left(\int_{\mathbb{R}} x^2 |\psi(x)|^2 dx\right) \cdot \left(\int_{\mathbb{R}} \xi^2 |\hat{\psi}(\xi)|^2 d\xi\right) \geq \frac{1}{16\pi^2}$,

and equality holds if and only if $\psi(x) = A e^{-Bx^2}$ for $B > 0$ and $A^2 = \sqrt{\frac{2B}{\pi}}$.

(b) $\left(\int_{\mathbb{R}} (x - x_0)^2 |\psi(x)|^2 dx\right) \cdot \left(\int_{\mathbb{R}} (\xi - \xi_0)^2 |\hat{\psi}(\xi)|^2 d\xi\right) \geq \frac{1}{16\pi^2}$ $\forall x_0, \xi_0 \in \mathbb{R}$

Rem. 1:

a) $f(x) = |\psi(x)|^2$ is a prob. density f'n (pdf) (≥ 0, $\int = 1$)

$\tilde{f}(\xi) = |\hat{\psi}(\xi)|^2$ is a pdf $[\int |\hat{\psi}|^2 \overset{\text{Plancherel}}{=} \int |\psi|^2 = 1]$

b) Let rnd. var's $X \sim f$, $Z \sim \tilde{f}$, then

Thm 2 (a) $\iff$ $E(X^2) \cdot E(Z^2) \geq \frac{1}{16\pi^2}$

c) Let $x_0 = E(X) = \int x f(x) dx$, $\xi_0 = E(Z) = \int \xi \tilde{f}(\xi) d\xi$, then

Thm 2(b) $\iff$ $Var(X) \cdot Var(Z) \geq \frac{1}{16\pi^2}$

d) In quantum mech, ψ = wave f'n, f pdf of position, $\tilde{f}$ pdf "momentum" $p = 2\pi\hbar\xi$ (de Broglie), and

Thm 2(b) $\Rightarrow$ $\underset{\text{std. dev.}}{\Delta x \cdot \Delta p} = \sqrt{Var(X)} \sqrt{Var(\overset{P = 2\pi\hbar Z}{P})} \geq \sqrt{\frac{(2\pi\hbar)^2}{16\pi^2}} = \frac{\hbar}{2}$

"(uncertainty of position) $\cdot$ (uncertainty of momentum) $\geq \frac{\hbar}{2}$"

5.)

"$\overset{\text{momentum}}{\text{Porsition}}$ certain ⟺ $\overset{\text{position}}{\text{momentum}}$ uncertain"

Pf.:

b) From a) $\overset{\text{by}}{\vee}$ replacing $\psi(x)$ by $e^{-2\pi i x \xi_0}\psi(x+x_0)$

a) $$1 = \int_{\mathbb{R}} 1\cdot|\psi|^2 \overset{\text{i.b.p.}}{\underset{\psi\in S(\mathbb{R})}{=}} -\int_{\mathbb{R}} x \frac{d}{dx}\underset{\overset{\shortparallel}{\psi\cdot\overline{\psi}}}{|\psi(x)|^2}\,dx$$

$$= -\int_{\mathbb{R}}(x\psi'\overline{\psi} + x\psi\overline{\psi'})\,dx = (\psi', -x\psi)_2 + \overline{(\psi', -x\psi)_2}$$

$\Downarrow$ 2 × C-S

$$(*) \quad 1 \le 2\|x\psi\|_2\cdot\|\psi'\|_2$$

$$\overset{\text{Plancherel}}{\shortparallel} \; \psi\in S(\mathbb{R})$$

$$\|\underbrace{\widehat{\psi'}}_{\underset{\psi\in S(\mathbb{R})}{=2\pi i\xi\hat{\psi}(\xi)}}\|_2$$

$$= 4\pi\|x\psi\|_2\cdot\|\xi\hat{\psi}\|_2$$

$\Updownarrow$

Formula in a)

Obs: = in $(*)$ if and only if $\psi' = \overline{c}x\psi$ and $\overline{\psi'} = cx\overline{\psi}$

$\Longrightarrow \psi(x) = Ae^{\frac{1}{2}cx^2}$, $A\in\mathbb{C}$ and $c=\overline{c}\in\mathbb{R}$

$\psi\in S(\mathbb{R}) \iff B := -\frac{c}{2} < 0$ $\square$

$\int|\psi|^2 = 1 \underset{\text{chk.}}{\iff} A^2 = \sqrt{\frac{2B}{\pi}}$ $\square$